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XV.

ON THE USES AND TRANSFORMATIONS OF LINEAR ALGEBRA.

BY BENJAMIN PEIRCE.

Presented, May 11, 1875.

SOME definite interpretation of a linear algebra would, at first sight, appear indispensable to its successful application. Whereas it is a singular fact, and one quite consonant with the principles of sound logic, that its first and general use is mostly to be expected from its want of significance. The interpretation is a trammel to the use. Symbols are essential to comprehensive argument. The familiar proposition that all A is B, and all B is C, and therefore all A is C, is contracted in its domain by the substitution of significant words for the symbolic letters. The A, B, and C, are subject to no limitation for the purposes and validity of the proposition; they may represent not merely the actual, but also the ideal, the impossible as well as the possible. In Algebra, likewise, the letters are symbols which, passed through a machinery of argument in accordance with given laws, are developed into symbolic results under the name of formulas. When the formulas admit of intelligible interpretation, they are accessions to knowledge; but independently of their interpretation they are invaluable as symbolical expressions of thought. But the most noted instance is the symbol, called the impossible or imaginary, known also as the square root of minus one, and which, from a shadow of meaning attached to it, may be more definitely distinguished as the symbol of *semi-inversion*. This symbol is restricted to a precise signification as the representative of perpendicularity in quaternions, and this wonderful algebra of space is intimately dependent upon the special use of the symbol for its symmetry, elegance, and power. The immortal author of quaternions has shown that there are other significations which may attach to the symbol in other cases. But the strongest use of the symbol is to be found in its magical power of doubling the actual universe, and placing by its side an ideal universe, its exact counterpart, with which it can be compared and contrasted, and, by means of curiously connecting fibres, form with it an organic whole, from which modern analysis has developed her surpassing geometry. The letters or units of the linear algebras, or to use the better term proposed by Mr. Charles S. Peirce, the *vids* of these algebras, are fitted to perform a similar function each in its peculiar way. This is their primitive and perhaps will always be their principal use. It does not exclude the possibility of some special modes of interpretation, but, on the contrary, a higher philosophy, which believes in the capacity of the material universe for all expressions of human thought, will find, in the utility of the vids, an indication of their probable reality of interpretation. Doctor Hermann Hankel's alternate numbers, with Professor Clifford's applications to determinants, are a curious and interesting example of the possible advantage to be obtained from the new algebras. Doctor Spottiswoode in his fine, generous, and complete analysis of my own treatise before the London Mathematical Society in November of 1872, has regarded these numbers as quite different from the algebras discussed in my treatise, because they are neither linear nor limited. But there is no difficulty in reducing them to a linear form, and, indeed, my algebra (e_3) is the simplest case of Hankel's alternate numbers, and in any other case in which n is the number of the Hankel elements employed, the complete number of vids of the corresponding linear algebra is $2^n - 1$. The limited character of the algebras which I have investigated may be regarded as an accident of the mode of discussion. There is, however, a large number of unlimited algebras suggested by the investigations, and Hankel's numbers themselves would have been a natural generalization from the proposition of § 65 of my algebra.* Another class of unlimited algebras, which would readily occur from the inspection of those which are given, is that in which all the powers of a vid are adopted as independent vids, and the highest power may either be zero, or unity, or the vid itself, and the zero power of the fundamental vid, *i. e.*, unity itself may also be retained as a vid. But I desire to draw especial attention to that class, which is also unlimited, and for which, when it was laid before the mathematical society of London in January of 1870, Professor Clifford proposed the appropriate name of *quadrates*.

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* This remark is not intended as a foundation for a claim upon the Hankel numbers, which were published in 1867, three years prior to the publication of my own treatise. — B. P.

QUADRATES.

The best definition of quadrates is that proposed by Mr. Charles S. Peirce. If the letters A, B, C, &c., represent absolute quantities, differing in quality, the vids may represent the relations of these quantities, and may be written in the form

$$(A:A)\ (A:B)\ (A:C)\ \ldots\ (B:A)\ (B:B)\ \ldots\ (C:A), \text{\&c.}$$

subject to the equations

$$(A:B)\ (B:C) = (A:C)$$
$$(A:B)\ (C:D) = 0.$$

i. e. every product vanishes, in which the second letter of the multiplier differs from the first letter of the multiplicand, and, when these two letters are identical, both are omitted, and the product is the vid which is compounded of the remaining letters which retain their relative position.

Mr. Peirce has shown by a simple logical argument that the quadrate is the legitimate form of a complete linear algebra, and that all the forms of the algebras given by me must be imperfect quadrates, and has confirmed this conclusion by actual investigation and reduction. His investigations do not however dispense with the analysis, by which the independent forms have been deduced in my treatise, but they seem to throw much light upon their probable use.

UNITY.

The sum of the vids $(A:A)$, $(B:B)$, $(C:C)$, &c., extended so as to include all the letters which represent absolute quantities in a given algebra, whether it be a complete or an incomplete quadrate, has the peculiar character of being idempotent, and of leaving any factor unchanged with which it is combined as multiplier or multiplicand. This is the distinguishing property of unity, so that this combination of the vids can be regarded as unity, and may be introduced as such and called the *vid of unity*. There is no other combination which possesses this property.

But either of the vids $(A:A)$, $(B:B)$, &c., or the sum of any of these vids is idempotent. There are many other idempotent combinations, such as

$$(A:A) + x\ (A:B),\ y\ (A:B) + (B:B),$$
$$\tfrac{1}{2}\ (A:A) + \tfrac{1}{2}\ (A:B) + \tfrac{1}{2}\ (B:A) + \tfrac{1}{2}\ (B:B),$$

which may deserve consideration in making transformations of an algebra preparatory to its application.

INVERSION.

A vid, which differs from unity, but of which the square is equal to unity, may be called a *vid of inversion.* For such a vid when applied to some other combination transforms it; but, whatever the transformation, a repetition of the application restores the combination to its primitive form. A very general form of a vid of inversion is

$$(A:A) \pm (B:B) \pm (C:C) \pm \text{\&c.},$$

in which each doubtful sign corresponds to two cases, except that at least one of the signs must be negative. The negative of unity might also be regarded as a symbol of inversion, but cannot take the place of an independent vid. Besides the above vids of inversion, others may be formed by adding to either of them a vid consisting of two different letters, which correspond to two of the one-lettered vids of different signs; and this additional vid may have any numerical coefficient whatever. Thus

$$(A:A) + (B:B) - (C:C) + x\,(A:C) + y\,(B:C)$$

is a vid of inversion.

The new vid which Professor Clifford has introduced into his biquaternions is a vid of inversion.

SEMI-INVERSION.

A vid of which the square is a vid of inversion, is a *vid of semi-inversion.* A very general form of a vid of semi-inversion is

$$(A:A) \pm (B:B) \pm \sqrt{-1}\,(C:C) \pm \text{\&c.}$$

in which one or more of the terms $(A:A)$, $(B:B)$, &c., have $\sqrt{-1}$ for a coefficient. The combination

$$(A:A) \pm \sqrt{-1}\,(B:B) + \tfrac{1}{2}(m \pm \sqrt{-1})\,(A:B)$$

is also a vid of semi-inversion. With the exception of unity, all the vids of Hamilton's quaternions are vids of semi-inversion.

THE USE OF COMMUTATIVE ALGEBRAS.

Commutative algebras are especially applicable to the integration of differential equations of the first degree with constant coefficients. If i, j, k, &c., are the vids of such an algebra, while x, y, z, &c., are independent variables, it is easy to show that a solution may have the form $F\,(xi + yj + zk + \text{\&c.})$, in which F is an arbitrary function, and i, j, k, &c., are connected by some simple equation. This solution can be developed into the form

$$F(xi + yj + zk + \&c.) = Mi + Nj + Pk + \&c.$$

in which M, N, P, &c., will be functions of x, y, z, &c., and each of them is a solution of the given equation. Thus in the case of Laplace's equation for the potential of attracting masses, the vids must satisfy the equation

$$i^2 + j^2 + k^2 = 0.$$

The algebra (a^3) of which the multiplication table is

	i	j	k
i	i	j	k
j	j	k	0
k	k	0	0

may be used for this case. Combinations i_1, j_1, k_1 of these vids can be found which satisfy the equation

$$i_1^2 + j_1^2 + k_1^2 = 0,$$

and if the functional solution

$$F(xi_1 + yj_1 + zk_1)$$

is developed into the form of the original vids

$$Mi + Nj + Pk.$$

M, N, and P will be independent solutions, of such a kind that the surfaces for which N and P are constant will be perpendicular to that for which M is constant, which is of great importance in the problems of electricity.

THE USE OF MIXED ALGEBRAS.

It is quite important to know the various kinds of pure algebra in making a selection for special use, but mixed algebras can also be used with advantage in certain cases. Thus in Professor Clifford's biquaternions of which he has demonstrated the great value, other vids can be substituted for unity, and his new vid, namely their half sum and half difference, and each of the original vids of the quaternions, can be multiplied by these, giving us two sets of vids, each of which will constitute an independent quadruple algebra of the same form with quaternions. Thus if i, j, k, are the primitive quaternion vids and w the new vid. Let

$$\begin{aligned} \alpha_1 &= \tfrac{1}{2}(1+w). & \alpha_2 &= \tfrac{1}{2}(1-w). \\ i_1 &= \alpha_1 i. & i_2 &= \alpha_2 i. \\ j_1 &= \alpha_1 j. & j_2 &= \alpha_2 j. \\ k_1 &= \alpha_1 k. & k_2 &= \alpha_2 k. \end{aligned}$$

Then since

$$\begin{aligned} \alpha_1^2 &= \alpha_1. & \alpha^2_2 &= \alpha_2. \\ i_1^2 = j_1^2 = k_1^2 &= -\alpha_1. & i_2^2 = j_2^2 = k_2^2 &= -\alpha_1. \\ i_1 j_1 = k_1 &= -j_1 i_1. & i_2 j_2 = k_2 &= -j_2 i_2. \\ j_1 k_1 = i_1 &= -k_1 j_1. & j_2 k_2 = i_2 &= -k_2 j_2. \\ k_1 i_1 = j_1 &= -i_1 k_1. & k_2 i_2 = j_2 &= -i_2 k_2. \end{aligned}$$

$$\alpha_1 \alpha_2 = 0 = \alpha_2 \alpha_1.$$

$$M_1 N_2 = 0 = N_2 M_1.$$

in which M_1 denotes any combination of the vids of the first algebra, and N_2 any combination of those of the second algebra. It may perhaps be claimed that these algebras are not independent, because the sum of the vids α_1 and α_2 is absolute unity. This, however, should be regarded as a fact of interpretation which is not apparent in the defining equations of the algebras.